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On a trajectory isomorphism of the Kowalevski gyrostat and the Clebsch problem

I V Komarov and A V Tsiganov

V A Fock Institute of Physics, St Petersburg State University, St Petersburg, Russia

E-mail: igor.komarov@pobox.phys.spbu.ru

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Abstract

For the Kowalevski gyrostat, a change of variables similar to that for the Kowalevski top is done. We establish one-to-one correspondence between solutions of the Kowalevski gyrostat and the Clebsch system and demonstrate that Kowalevski variables for the gyrostat practically coincide with elliptic coordinates on a sphere for the Clebsch case.

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1. Introduction

The aim of this paper is to extend the Kowalevski treatment of the top [14] to the gyrostat and to incorporate the gyrostat into the Heine–Horozov scheme [5] that reproduces separation of variables for the top.

Let two vectors $\boldsymbol{J}$ and $\boldsymbol{x}$ be coordinates on the phase space M. As a Poisson manifold M is identified with Euclidean algebra $e(3)^*$ with the Lie–Poisson brackets

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \qquad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \qquad \{x_i, x_j\} = 0, \tag{1.1}$$

where ε_{ijk} is the totally skew-symmetric tensor. These brackets have two Casimir functions

$$A = \boldsymbol{x}^2 \equiv \sum_{k=1}^{3} x_k^2, \qquad B = (\boldsymbol{x} \cdot \boldsymbol{J}) \equiv \sum_{k=1}^{3} x_k J_k.$$

Fixing their values one gets a generic symplectic leaf of $e(3)$

$$\mathcal{O}_{ab}: \quad \{\boldsymbol{x}, \boldsymbol{J} : \quad A = a, \quad B = b\},$$

which is a four-dimensional symplectic manifold.

The equations of motion on $e(3)^*$ are given by the customary Euler–Poisson equations

$$X: \quad \dot{\boldsymbol{J}} = \boldsymbol{J} \times \frac{\partial H}{\partial \boldsymbol{J}} + \boldsymbol{x} \times \frac{\partial H}{\partial \boldsymbol{x}}, \qquad \dot{\boldsymbol{x}} = \boldsymbol{x} \times \frac{\partial H}{\partial \boldsymbol{J}}, \tag{1.2}$$

where $\boldsymbol{x} \times \boldsymbol{z}$ means cross product of two vectors.

The Hamilton function for the original Kowalevski top is given by

$$H_{\text{top}} = \tfrac{1}{2}\left(J_1^2 + J_2^2 + 2J_3^2\right) + cx_1, \qquad c \in \mathbb{C}.$$

This Hamiltonian and additional integral of motion $K_{\text{top}} = \xi_1 \cdot \xi_2$ are in involution and define a moment map whose fibres are Liouville tori in $\mathcal{E}_{ab}$. Here

$$\xi_1 = z_1^2 - 2c(x_1 + \mathrm{i}x_2), \qquad \xi_2 = z_2^2 - 2c(x_1 - \mathrm{i}x_2) \tag{1.3}$$

and

$$z_1 = J_1 + \mathrm{i}J_2, \qquad z_2 = J_1 - \mathrm{i}J_2. \tag{1.4}$$

The Kowalevski gyrostat [8, 18] is an integrable extension of the corresponding top defined by the following constants of motion:

$$\begin{aligned} H &= H_{\text{top}} - \lambda J_3 = \tfrac{1}{2}\left(J_1^2 + J_2^2 + 2J_3^2 - 2\lambda J_3\right) + cx_1, \\ K &= \xi_1\xi_2 + 4\lambda((J_3 - \lambda)z_1z_2 - (z_1 + z_2)cx_3) \end{aligned} \tag{1.5}$$

in involution $\{H, K\} = 0$. The gyrostat generalization of the Kowalevski top is essential because the corresponding additional terms in the Hamiltonian mimic quantum corrections to the top [8].

2. The Kowalevski gyrostat in the Kowalevski s-variables

We will introduce variables $s_{1,2}$ for the Kowalevski gyrostat step by step following the original papers [14] and [13] where the separation of variables for the top was constructed.

First we made a transition from initial variables to new variables $\xi_{1,2}$ (1.3), $z_{1,2}$ (1.4) and organize four constants of motion in the following matrix identity:

$$\begin{aligned} \begin{pmatrix} 4H & 4cB \\ 4cB & 4c^2A - K \end{pmatrix} &= 4\begin{pmatrix} J_3^2 & cx_3J_3 \\ cx_3J_3 & c^2x_3^2 \end{pmatrix} - 4\lambda\begin{pmatrix} J_3 & 0 \\ 0 & z_1z_2(J_3 - \lambda) - c(z_1 + z_2)x_3 \end{pmatrix} \\ &\quad + \begin{pmatrix} (z_1 + z_2)^2 & z_1z_2(z_1 + z_2) \\ z_1z_2(z_1 + z_2) & z_1^2z_2^2 \end{pmatrix} - \begin{pmatrix} \xi_1 + \xi_2 & \xi_1z_2 + \xi_2z_1 \\ \xi_1z_2 + \xi_2z_1 & \xi_1z_2^2 + \xi_2z_1^2 \end{pmatrix}. \end{aligned} \tag{2.1}$$

The second step consists of exclusion of two variables x_3 and J_3 using velocities $\dot{z}_i = \{H, z_i\}$

$$x_3 = \frac{\mathrm{i}}{c}\,\frac{\dot{z}_1z_2 + z_1\dot{z}_2}{z_1 - z_2}, \qquad J_3 = \frac{\mathrm{i}(\dot{z}_2 + \dot{z}_1)}{(z_1 - z_2)} + \lambda. \tag{2.2}$$

Similarity transform $U^t(\cdot)U$ of the both sides of (2.1) with auxiliary matrix U

$$U = \begin{pmatrix} z_1 & z_2 \\ -1 & -1 \end{pmatrix}, \qquad U^t = \begin{pmatrix} z_1 & -1 \\ z_2 & -1 \end{pmatrix} \tag{2.3}$$

brings us to the following matrix identity for the gyrostat:

$$\begin{aligned} &4\begin{pmatrix} \dot{z}_1^2 & -\dot{z}_1\dot{z}_2 \\ -\dot{z}_1\dot{z}_2 & \dot{z}_2^2 \end{pmatrix} + 4\mathrm{i}\lambda(z_1 - z_2)\begin{pmatrix} \dot{z}_1 & 0 \\ 0 & \dot{z}_2 \end{pmatrix} + (z_1 - z_2)^2\begin{pmatrix} \xi_1 & -2H \\ -2H & \xi_2 \end{pmatrix} \\ &\quad - \begin{pmatrix} R(z_1, z_1) & R(z_1, z_2) \\ R(z_1, z_2) & R(z_2, z_2) \end{pmatrix} = 0. \end{aligned} \tag{2.4}$$

Here

$$R(z_1, z_2) = z_1^2z_2^2 - 2H\left(z_1^2 + z_2^2\right) - 4cB(z_1 + z_2) - 4c^2A + K. \tag{2.5}$$

The diagonal entries of identity (2.4) allows us to express variables $\xi_{1,2}$ as

$$\xi_k = \frac{4\mathrm{i}\lambda\dot{z}_k}{z_1 - z_2} - \frac{4\dot{z}_k^2 - R(z_k, z_k)}{(z_1 - z_2)^2}, \qquad k = 1, 2, \tag{2.6}$$

and one gets integrals H and K (1.5) in terms of biquadratic polynomial R (2.5) and two pairs of Lagrangian variables $z_{1,2}$ and $\dot{z}_{1,2}$

$$H = -\frac{4\dot{z}_1\dot{z}_2 + R(z_1, z_2)}{2(z_1 - z_2)^2}, \tag{2.7}$$

$$K = -\frac{16\dot{z}_1\dot{z}_2}{(z_1 - z_2)^2}\lambda^2 - 4\mathrm{i}\lambda\left(\dot{z}_1\frac{\partial}{\partial z_1} - \dot{z}_2\frac{\partial}{\partial z_2}\right)\frac{R(z_1, z_2)}{(z_1 - z_2)^2} + \frac{\big(4\dot{z}_1^2 - R(z_1, z_1)\big)\big(4\dot{z}_2^2 - R(z_2, z_2)\big)}{(z_1 - z_2)^4}. \tag{2.8}$$

The unexpected appearance of differential operator in this relation is a main qualitative difference of the gyrostat from the top.

On the level surface of integrals of motion

$$\Sigma = \{A = a, B = b, H = h, K = k\} \tag{2.9}$$

relations (2.7) and (2.8)

$$\Phi_{1,2}(z_1, z_2, \dot{z}_1, \dot{z}_2, A, B, H, K)|_\Sigma = 0 \tag{2.10}$$

can be considered as equations of motion determining a two-dimensional dynamical system.

Lemma 1. *On the level surface of integrals of motion Σ there is one-to-one correspondence between solutions $x_i(t)$ and $J_i(t)$ of the Kowalevski gyrostat problem and solutions $z_{1,2}(t)$ of this dynamical system.*

The map $\{\boldsymbol{x}, \boldsymbol{J}\} \to \{z_{1,2}, \dot{z}_{1,2}\}$ is given by (1.4). The inverse map consists of relations (2.2) and combination of the mapping $(z_{1,2}, \dot{z}_{1,2}) \to (z_{1,2}, \xi_{1,2})$ (2.6) with the following equations:

$$x_1 = \frac{1}{4c}\big(z_1^2 + z_2^2 - \xi_1 - \xi_2\big), \qquad J_1 = \frac{1}{2}(z_1 + z_2)$$

$$x_2 = -\frac{\mathrm{i}}{4c}\big(z_1^2 - z_2^2 - \xi_1 + \xi_2\big), \qquad J_2 = \frac{\mathrm{i}}{2}(z_1 - z_2).$$

So, instead of the initial equations of motion (1.2) we can solve the auxiliary dynamical equations (2.10). Unfortunately variables $z_{1,2}$ do not commute $\{z_1, z_2\} \neq 0$, so one has to look for a more convenient parametrization.

Associated with the fourth degree polynomials $R(z_k, z_k)$ (2.5)

$$R(z_k, z_k) = a_0 z_k^4 + 4a_1 z_k^3 + 6a_2 z_k^2 + 4a_3 z_k + a_4, \qquad a_i \in \mathbb{R}$$

the differential equations

$$\frac{\dot{z}_1}{\sqrt{R(z_1, z_1)}} = \pm\frac{\dot{z}_2}{\sqrt{R(z_2, z_2)}}, \tag{2.11}$$

originally appeared in the Euler studies of equation of lemniscate and invariance of the corresponding elliptic integrals [3]. In particular Euler proved that equations (2.11) have an algebraic integral

$$\mathcal{E}(z_1, z_2, s) = (z_1 - z_2)^2 s^2 - R(z_1, z_2)\, s + W = 0, \tag{2.12}$$

where $R(z_1, z_2)$ is a mixed biquadratic form similar to (2.5)

$$R(z_1, z_2) = a_0 z_1^2 z_2^2 + 2a_1 z_1 z_2 (z_1 + z_2) + 3a_2\left(z_1^2 + z_2^2\right) + 2a_3(z_1 + z_2) + a_4,$$

and

$$W = \frac{R(z_1, z_2)^2 - R(z_1, z_1)R(z_2, z_2)}{4(z_1 - z_2)^2}.$$

In algebro-geometric terms [17], Euler studied automorphisms $(u_1, z_1) \to (u_2, z_2)$ of the algebraic curve of genus one

$$\mathcal{C}: \quad u^2 = R(z, z),$$

which change a sign of the corresponding holomorphic form $\mathrm{d}z/u \to \pm\,\mathrm{d}z/u$. Thus every algebraic curve of genus one is isomorphic to a complex torus (cubic elliptic curve), which is equivalent to a Jacobian of $\mathcal{C}$. These automorphisms are parametrized by points of a smooth elliptic curve

$$\Gamma: \quad \eta^2 = P_3(s), \qquad P_3(s) = 4s^3 + g_1 s^2 + g_2 s + g_3,$$

where g_k are functions on initial parameters $a_0, \ldots, a_4$. According to Weil [17], if $O_k = (u_k, z_k), k = 1, 2$, are two points of $\mathcal{C}$ and $N_k = (\eta_k, s_k), k = 1, 2$, denote two points of Γ related by $O_1 = N_1 + N_2$ and $O_2 = N_1 - N_2$ then

$$\frac{\mathrm{d}z_1}{u_1} + \frac{\mathrm{d}z_2}{u_2} = \frac{\mathrm{d}s_1}{\eta_1}, \qquad \frac{\mathrm{d}z_1}{u_1} - \frac{\mathrm{d}z_2}{u_2} = \frac{\mathrm{d}s_2}{\eta_2}.$$

This is an infinitesimal version of the Weil interpretation of the Euler results. These results are independent of the choice of affine coordinates (u, z) and (η, s) of the curves $\mathcal{C}$ and Γ, respectively.

The third step of Kowalevski in [14] is an introduction of her famous variables $s_{1,2}$

$$s_{1,2} = \frac{R(z_1, z_2) \pm \sqrt{R(z_1, z_1)R(z_2, z_2)}}{2(z_1 - z_2)^2}, \tag{2.13}$$

which are transcendental integrals of the corresponding Euler equations (2.11) [3]. We use her definition of $s_{1,2}$ substituting integrals of motion of the top with that of the gyrostat. By definition (1.4) in physical domain variables $s_{1,2}$ are real and satisfy the inequality

$$s_1 \leqslant s_2. \tag{2.14}$$

Lemma 2. *On the level surface of integrals of motion Σ we have one-to-one correspondence between variables $z_{1,2}$ (1.4) and $s_{1,2}$ (2.13).*

The algebro-geometric proof may be found in [17].

The variables $s_{1,2}$ are eigenvalues of an auxiliary spectral problem

$$\begin{pmatrix} R(z_1, z_1) & R(z_1, z_2) \\ R(z_1, z_2) & R(z_2, z_2) \end{pmatrix} \Psi = 2s\sigma_1 (z_1 - z_2)^2 \Psi, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.15}$$

that is naturally extracted from (2.4). Its characteristic polynomial,

$$\mathcal{E}(s) = (z_1 - z_2)^2 (s - s_1)(s - s_2), \tag{2.16}$$

coincides with the Euler algebraic integral (2.12). The matrix of eigenfunctions Ψ of the spectral problem (2.15) reads

$$\Psi = \begin{pmatrix} \dfrac{1}{\sqrt{R(z_1, z_1)}} & \dfrac{1}{\sqrt{R(z_2, z_2)}} \\ -\dfrac{1}{\sqrt{R(z_1, z_1)}} & \dfrac{1}{\sqrt{R(z_2, z_2)}} \end{pmatrix}. \tag{2.17}$$

The idea of Kowalevski to pass to new variables $s_{1,2}$ is appeared to be very fruitful in her treatment of the top. For the Kowalevski gyrostat as well as for the top these variables have the following main property:

Theorem 1. *Functions $s_{1,2}$ (2.13) are Poisson commute $\{s_1, s_2\} = 0$.*

Proof. For $\lambda = 0$ the straightforward proof may be founded in [4, 9]. For $\lambda \neq 0$ this unexpected and crucial observation was obtained by direct calculation of the Poisson brackets. □

Theorem 2. *On the level surface of integrals of motion (2.9) variables $s_{1,2}$ (2.13) satisfy the following dynamical equations:*

$$\left((s_1 - s_2)^2 \dot{s}_k^2 + \lambda\sqrt{-\varphi_1\varphi_2}\,\dot{s}_k - \beta_k\varphi_k\right)^2 + \lambda^2 \left(\dot{s}_k^2 + \frac{(2H + s_1 + s_2)\varphi_k}{s_1 - s_2}\right)\varphi_k^2 = 0, \qquad k = 1, 2, \tag{2.18}$$

where β_k is given by

$$\beta_k = (2H + s_1 + s_2)\lambda^2 + s_k^2 + 2Hs_k + H^2 - \frac{K}{4}. \tag{2.19}$$

Proof. Function $\mathcal{E}(z_1, z_2, s)$ (2.12), (2.16) is a quadratic polynomial with respect to any of its three arguments z_1, z_2, s. Its partial derivatives with respect to one of the variables are discriminants of the corresponding quadratic equations. Squares of its partial derivatives with respect to one of the variables are factorized into functions of the other two

$$\left(\frac{\partial\mathcal{E}}{\partial s}\right)^2 = R(z_1, z_1)R(z_2, z_2), \qquad \left(\frac{\partial\mathcal{E}}{\partial z_k}\right)^2 = R(z_k, z_k)P_3(s), \qquad k = 1, 2.$$

Here polynomial $P_3(s)$ is given by

$$P_3(s) = 4s^3 - 8Hs^2 + 4H^2 s - Ks + 4c^2 As + 4c^2 B \tag{2.20}$$

Because the complete differential of $\mathcal{E}(s, z_1, z_2)$ (2.12) is zero

$$\frac{\partial\mathcal{E}}{\partial s}\,\mathrm{d}s + \frac{\partial\mathcal{E}}{\partial z_1}\,\mathrm{d}z_1 + \frac{\partial\mathcal{E}}{\partial z_2}\,\mathrm{d}z_2 = 0,$$

one gets relations between the differentials of the variables of both types

$$\frac{\mathrm{d}s_{1,2}}{\sqrt{P_3(s_{1,2})}} = \frac{\mathrm{d}z_1}{\sqrt{R(z_1, z_1)}} \pm \frac{\mathrm{d}z_2}{\sqrt{R(z_2, z_2)}}. \tag{2.21}$$

In matrix form the relations for velocities look like

$$\begin{pmatrix} \frac{\dot{s}_1}{\sqrt{\varphi_1}} \\ \frac{\dot{s}_2}{\sqrt{\varphi_2}} \end{pmatrix} = \Psi \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix}, \tag{2.22}$$

where we denoted for brevity

$$\varphi_k \equiv P_3(s_k). \tag{2.23}$$

Signs at square roots in (2.21), (2.22) are compatible with definition of $s_{1,2}$ (2.13) and Ψ (2.17).

Using (2.5)–(2.22) we can express integrals of motion H (2.7) and K (2.8) in terms of cubic polynomial $P_3(s)$, variables $s_{1,2}$ and their velocities $\dot{s}_{1,2}$

$$H = \frac{s_1 - s_2}{2}\left(\frac{\dot{s}_1^2}{\varphi_1} - \frac{\dot{s}_2^2}{\varphi_2}\right) - \frac{s_1 + s_2}{2}, \tag{2.24}$$

$$\frac{K}{4} = (2H + s_1 + s_2)\lambda^2 - \lambda\sqrt{-\varphi_1\varphi_2}\left(\frac{\dot s_1}{\varphi_1} + \frac{\dot s_2}{\varphi_2}\right) + (s_1 - s_2)\left(\frac{s_2\dot s_1^2}{\varphi_1} - \frac{s_1\dot s_2^2}{\varphi_2}\right) - s_1 s_2 + H^2. \tag{2.25}$$

Here the Hamiltonian H and coefficients of integral K at even powers of the gyrostatic parameter λ are easy calculated using definitions (2.13) and (2.22) only. For the linear in λ term in $K = K_2\lambda^2 + K_1\lambda + K_0$ one gets at first

$$K_1 = -4\mathrm{i}\frac{\dot s_1}{\sqrt{\varphi_1}}\left(\sqrt{R(z_1,z_1)}\frac{\partial}{\partial z_1} - \sqrt{R(z_2,z_2)}\frac{\partial}{\partial z_2}\right)\frac{R(z_1,z_2)}{(z_1-z_2)^2}$$
$$\qquad - 4\mathrm{i}\frac{\dot s_2}{\sqrt{\varphi_2}}\left(\sqrt{R(z_1,z_1)}\frac{\partial}{\partial z_1} + \sqrt{R(z_2,z_2)}\frac{\partial}{\partial z_2}\right)\frac{R(z_1,z_2)}{(z_1-z_2)^2}.$$

Due to the inverse of (2.21) one converts derivatives $\partial/\partial z_{1,2}$ to $\partial/\partial s_{1,2}$

$$K_1 = -4\mathrm{i}\left(\dot s_1\frac{\sqrt{\varphi_2}}{\sqrt{\varphi_1}}\frac{\partial}{\partial s_1} + \dot s_2\frac{\sqrt{\varphi_1}}{\sqrt{\varphi_2}}\frac{\partial}{\partial s_2}\right)\frac{R(z_1,z_2)}{(z_1-z_2)^2}.$$

Keeping in mind from definition (2.13) that one gets $s_1 + s_2 = R(z_1,z_2)/(z_1-z_2)^2$, and including i into the square root we obtain finally

$$K_1 = -4\left(\dot s_1\sqrt{\frac{-\varphi_2}{\varphi_1}} + \dot s_2\sqrt{\frac{-\varphi_1}{\varphi_2}}\right).$$

Equations (2.24) and (2.25) have the form

$$\widetilde\Phi_{1,2}(s_1, s_2, \dot s_1, \dot s_2, A, B, H, K)|_\Sigma = 0,$$

and depend on the commuting variables $s_{1,2}$, their velocities $\dot s_{1,2}$ and integrals of motion only. Excluding one of the velocities we obtain two equations (2.18) of fourth order in $\dot s_k$. □

Equations of motion (2.18) determine some two-dimensional dynamical system on Σ.

Lemma 3. *On the level of integrals of motion Σ (2.9) there is one-to-one correspondence between solutions $x_i(t)$ and $J_i(t)$ of the Kowalevski gyrostat problem and solutions $s_1(t)$ and $s_2(t)$ of the dynamical equations (2.18).*

The proof consists of a combination of lemmas 1 and 2.

Remark. At $\lambda = 0$, equations (2.18) are reduced to the Kowalevski top equations [14, 13]

$$(-1)^k(s_1 - s_2)\dot s_k = \sqrt{P_5(s_k)}, \qquad k = 1, 2,$$

which admit integration on Σ (2.9) by the Jacobi inversion theorem. Here $P_5(s) = P_3(s)P_2(s)$ is a fifth-order polynomial, $P_3(s)$ is from (2.20) and

$$P_2(s) = s^2 + 2Hs + H^2 - \frac{K}{4}$$

is a limiting value of β_k (2.19), $P_2(s_k) = \beta_k|_{\lambda=0}$.

At $\lambda = 0$ transformation $\{s_{1,2}, \dot s_{1,2}\}|_\Sigma \to \{J_i, x_i\}$ is discussed in [14, 13] in detail.

3. The Kowalevski gyrostat and the Clebsch system

Let two vectors $\boldsymbol{l}$ and $\boldsymbol{p}$ be coordinates on the phase space $\mathcal{M}$. As a Poisson manifold $\mathcal{M}$ is identified with the algebra $e(3)^*$ equipped with brackets

$$\{l_i, l_j\} = \varepsilon_{ijk} l_k, \qquad \{l_i, p_j\} = \varepsilon_{ijk} p_k, \qquad \{p_i, p_j\} = 0. \tag{3.1}$$

These brackets respect two Casimir elements

$$\mathcal{A} = (\boldsymbol{p}, \boldsymbol{p}), \qquad \mathcal{B} = (\boldsymbol{p}, \boldsymbol{l}). \tag{3.2}$$

The following integrable case for the Kirchhoff equations on $e(3)$ was found by Clebsch [2]

$$\mathcal{X}: \quad \dot{\boldsymbol{l}} = \boldsymbol{p} \times \mathbf{Q}\boldsymbol{p}, \qquad \dot{\boldsymbol{p}} = \boldsymbol{p} \times \boldsymbol{l}. \tag{3.3}$$

Here $\mathbf{Q}$ is a constant symmetric matrix, $\det \mathbf{Q} \neq 0$. Equations of motion (3.3) are generated by brackets (3.1) and the Hamilton function

$$\mathcal{H} = \tfrac{1}{2}\boldsymbol{l}^2 + \tfrac{1}{2}(\mathbf{Q}\boldsymbol{p}, \boldsymbol{p}). \tag{3.4}$$

The second integral of motion reads as

$$\mathcal{K} = (\mathbf{Q}\boldsymbol{l}, \boldsymbol{l}) - (\mathbf{Q}^{\vee}\boldsymbol{p}, \boldsymbol{p}), \tag{3.5}$$

where $\mathbf{Q}^{\vee}$ stands for adjoint matrix, i.e. cofactor matrix. In our case it reads $\mathbf{Q}^{\vee} = (\det \mathbf{Q})\mathbf{Q}^{-1}$.

If $\mathcal{B} = 0$, flow (3.3) is equivalent to that of the Neumann system describing the motion of a mass point on the sphere $\boldsymbol{p}^2 = \mathcal{A}$ under the influence of the force $-\mathbf{Q}\boldsymbol{p}$.

3.1. The Clebsch system in elliptic coordinates

Minkowski [15] identified the Clebsch system with the Jacobi problem of geodesic motion on ellipsoid for which elliptic coordinates $u_{1,2}$ were introduced by Jacobi. In 1895 Kobb started the integration procedure in the Euler angles and passed to variables $\xi = \tan(\theta/2)$, $\nu = \tan(\phi/2)$, which are equivalent to variables $u_{1,2}$ [7]. In 1959 Kharlamova [6] used elliptic coordinates directly for integration of the second flow of the Clebsch system associated with $\mathcal{K}$.

In order to explain the method proposed in [6, 7] we reproduce some simple formulae. Using equations of motion (3.3) and the Casimir elements (3.2) we express angular momenta $\boldsymbol{l}$ via Lagrangian variables $\boldsymbol{p}, \dot{\boldsymbol{p}}$

$$\boldsymbol{l} = \frac{1}{\mathcal{A}}(\mathcal{B}\boldsymbol{p} + \dot{\boldsymbol{p}} \times \boldsymbol{p}). \tag{3.6}$$

Then we introduce variables $u_{1,2}$ as roots of the following function:

$$e(\mu) = (\mu - u_1)(\mu - u_2) = \mu^2 + \left(\frac{(\mathbf{Q}\boldsymbol{p}, \boldsymbol{p})}{\mathcal{A}} - \operatorname{tr}\mathbf{Q}\right)\mu + \frac{(\mathbf{Q}^{\vee}\boldsymbol{p}, \boldsymbol{p})}{\mathcal{A}}. \tag{3.7}$$

Variables $u_{1,2}$ are real and we can always impose

$$u_1 \leqslant u_2 \tag{3.8}$$

similar to (2.14). Substituting $u_{1,2}$ and their velocities $\dot{u}_{1,2}$ into (3.4) and (3.5) one gets the Hamilton function

$$\mathcal{H} = \frac{u_1 - u_2}{2}\left(\frac{\dot{u}_1^2}{\varphi_1} - \frac{\dot{u}_2^2}{\varphi_2}\right) + \frac{\mathcal{B}^2}{2\mathcal{A}} + \frac{1}{2}(\operatorname{tr}\mathbf{Q} - u_1 - u_2)\mathcal{A},$$

and the second integral of motion

$$\mathcal{K} = (u_1 - u_2)\left(\frac{u_2\dot{u}_1^2}{\varphi_1} - \frac{u_1\dot{u}_2^2}{\varphi_2}\right) - \frac{\mathcal{B}}{\sqrt{\mathcal{A}}}\left(\dot{u}_1\sqrt{-\frac{\varphi_2}{\varphi_1}} + \dot{u}_2\sqrt{-\frac{\varphi_1}{\varphi_2}}\right)$$
$$+ \frac{\mathcal{B}^2}{\mathcal{A}}(\operatorname{tr}\mathbf{Q} - u_1 - u_2) - u_1u_2\mathcal{A}$$

in terms of variables $u_{1,2}$, their velocities $\dot{u}_{1,2}$ and the cubic polynomial

$$\varphi_k = 4\det(\mathbf{Q} - u_k\mathbf{I}).$$

Below this polynomial will be identified with the cubic polynomial φ_k (2.23) for the Kowalevski gyrostat for which we used the same notation.

Excluding one of the velocities from these equations we obtain two equations of fourth degree in each of the velocities depending on both variables u_1 and u_2

$$\big(\mathcal{A}(u_1-u_2)^2\dot{u}_k^2 + \mathcal{B}\sqrt{-\mathcal{A}\varphi_1\varphi_2}\,\dot{u}_k + \beta_k\varphi_k\big)^2 + \mathcal{B}^2\left(\mathcal{A}\dot{u}_k^2 + \frac{(\mathcal{A}^2(u_1+u_2-\operatorname{tr}\mathbf{Q}) + 2\mathcal{A}\mathcal{H} - \mathcal{B}^2)\varphi_k}{u_1-u_2}\right)\varphi_k^2 = 0. \tag{3.9}$$

Here β_k is a cubic polynomial also depending on u_1 and u_2

$$\beta_k = \mathcal{B}^2(u_1+u_2+u_k-\operatorname{tr}\mathbf{Q}) - \mathcal{A}\big(\mathcal{A}u_k^2 + (2\mathcal{H} - \mathcal{A}\operatorname{tr}\mathbf{Q})u_k - \mathcal{K}\big).$$

Equations (3.9) were solved in quadratures in [6, 7]. Obviously $u_{1,2}$ are not the separated variables for the Clebsch system. We think that they will be helpful in construction of separation of variables for the Clebsch system.

Lemma 4. *There is one-to-one correspondence between solutions of the Kirchhoff equations (3.3) in the Clebsch case and solutions of the dynamical equations (3.9).*

Proof. Variables $u_{1,2}$ as functions of p_j are given by (3.7). In order to construct inverse mapping let us make a suitable rotation

$$\tilde{\boldsymbol{p}} = V\boldsymbol{p}, \quad \tilde{\boldsymbol{l}} = V\boldsymbol{l}, \quad \mathbf{Q} \to \widetilde{\mathbf{Q}} = V\mathbf{Q}V^{-1} = \operatorname{diag}(a_1, a_2, a_3), \tag{3.10}$$

which diagonalize the matrix $\mathbf{Q}$. In this case coordinates $u_{1,2}$ coincide with elliptic coordinates on sphere $\boldsymbol{p}^2 = \mathcal{A}$ defined by

$$e(\mu) = (\mu-u_1)(\mu-u_2) = \frac{\det(\mathbf{Q}-\mu\mathbf{I})}{\mathcal{A}}\left(\frac{\tilde{p}_1^2}{\mu-a_1} + \frac{\tilde{p}_2^2}{\mu-a_2} + \frac{\tilde{p}_3^2}{\mu-a_3}\right). \tag{3.11}$$

The mapping of the elliptic coordinates $u_{1,2}$ and their velocities $\dot{u}_{1,2}$ to variables $\boldsymbol{p}$ and $\boldsymbol{l}$ is well studied [1]. Namely, substituting solutions of the equations (3.9) into

$$p_j = \sqrt{\mathcal{A}\frac{\prod_{k=1}^2(a_j-u_k)}{\prod_{i\neq j}^3(a_j-a_i)}}, \tag{3.12}$$

one gets vector $\tilde{\boldsymbol{p}}(t)$, which after inverse rotation with respect to (3.10) gives rise to solutions $\boldsymbol{p}(t)$ and $\boldsymbol{l}(t)$ (3.6) of the Clebsch system. □

3.2. *Mapping of the Kowalevski gyrostat flow onto the Clebsch flow*

The idea of the map Kowalevski top flow (1.2) onto the Neumann flow originally appeared in Heine and Horosov [5, 16] for the Kowalevski top and was extended to $so(4)$, $so(3,1)$ in [10].

Let us introduce the following complex vector-functions:

$$\boldsymbol{p} = \alpha\left(-\mathrm{i}\frac{J_1}{J_2}, \frac{J_1^2+J_2^2+1}{2J_2}, \mathrm{i}\frac{J_1^2+J_2^2-1}{2J_2}\right), \qquad \alpha \in \mathbb{R}, \tag{3.13}$$

and

$$\boldsymbol{l}_{\text{top}} = \left(-\mathrm{i}\,\frac{cx_3}{J_2}, \frac{2cx_3J_1 - J_3\big(J_1^2+J_2^2-1\big)}{2J_2}, \mathrm{i}\,\frac{2cx_3J_1 - J_3\big(J_1^2+J_2^2+1\big)}{2J_2}\right), \tag{3.14}$$

such that

$$\mathcal{A} = (\boldsymbol{p}, \boldsymbol{p}) = \alpha^2, \qquad \mathcal{B} = (\boldsymbol{p}, \boldsymbol{l}_{\text{top}}) = 0. \tag{3.15}$$

We permuted the first and the second entries in original vectors [5] to make the gyrostat formulae slightly more symmetric.

In order to describe mapping of the gyrostat flow (1.2) onto the Clebsch flow (3.3) we have to shift the vector $\boldsymbol{l}_{\text{top}}$ by the rule

$$\boldsymbol{l} = \boldsymbol{l}_{\text{top}} + \alpha^{-1}\lambda(\boldsymbol{p} + i\boldsymbol{k} \times \boldsymbol{p}), \tag{3.16}$$

where $\boldsymbol{k} = (1, 0, 0)$ is a unit vector. In comparison with (3.15) the scalar product of vectors $\boldsymbol{l}$ and $\boldsymbol{p}$ for the gyrostat becomes different from zero

$$\mathcal{B} = (\boldsymbol{p}, \boldsymbol{l}) = \alpha\lambda.$$

For the Kowalevski top and gyrostat variables $\boldsymbol{p}, \boldsymbol{l}_{\text{top}}$ and $\boldsymbol{p}, \boldsymbol{l}$ are coordinates on the different spaces $\mathcal{M}_{\text{top}}$ and $\mathcal{M}$ with different brackets (3.1) $\{\cdot, \cdot\}_{\text{top}}$ and $\{\cdot, \cdot\}$ forming two samples of $e(3)$ algebra (3.1). With respect to the top brackets $\{\cdot, \cdot\}_{\text{top}}$ the gyrostat variables $\boldsymbol{p}, \boldsymbol{l}$ form the central extension of $e(3)_{\text{top}}$ which is contracted to $e(3)_{\text{top}}$ in the limit $\lambda \to 0$.

Similar to the top [5] let us introduce symmetric matrix $\mathbf{Q}$ linearly depending on integrals of motion of the Kowalevski gyrostat and the Casimir elements on the initial algebra (1.1)

$$\mathbf{Q} = \alpha^{-2}\begin{pmatrix} -H & -\mathrm{i}cb & \mathrm{i}cb \\ -\mathrm{i}cb & -\frac{1}{4} + c^2\varkappa & \mathrm{i}\left(\frac{1}{4} + c^2\varkappa\right) \\ \mathrm{i}cb & \mathrm{i}\left(\frac{1}{4} + c^2\varkappa\right) & \frac{1}{4} - c^2\varkappa \end{pmatrix}, \qquad \varkappa = a - K/4c^2. \tag{3.17}$$

This matrix remains constant with respect to the dynamics of the Clebsch system on $\mathcal{M}$.

Theorem 3. *Let us identify $\mathcal{M}$ with M by the map $\{\boldsymbol{x}, \boldsymbol{J}\} \to \{\boldsymbol{p}, \boldsymbol{l}\}$ (3.13), (3.16) such that the Casimir elements are equal to*

$$A = a, \quad B = b, \quad \mathcal{A} = \alpha^2, \quad \mathcal{B} = \alpha\lambda. \tag{3.18}$$

If the matrix $\mathbf{Q}$ is given by (3.17), then

$$2\mathcal{H} = -H + \lambda^2, \qquad 4\alpha^2\mathcal{K} = K - 4\lambda^2 H, \tag{3.19}$$

and vector field X (1.2) for the Kowalevski gyrostat on M coincides with vector field $\mathcal{X}$ (3.3) for the Clebsch system on $\mathcal{M}$. A similar equality holds for the second commuting flows of the Kowalevski gyrostat and the Clebsch system.

The proof is straightforward.

3.3. Mapping of the Clebsch flow onto the Kowalevski gyrostat flow

Below without loss of generality we put $\alpha = |\boldsymbol{p}|^2 = 1$ in (3.18). In this case according to (3.19) values of integrals for the Kowalevski gyrostat and the Clebsch system are connected by the relations

$$H = -2\mathcal{H} + \lambda^2, \qquad K = 4\mathcal{K} - 8\lambda^2\mathcal{H} + 4\lambda^4. \tag{3.20}$$

Inserting $\boldsymbol{p}$ (3.13) and $\boldsymbol{l}$ (3.16) into the generating function $e(\mu)$ (3.7) of u-variables, one gets

$$e(\mu) = \frac{1}{(z_1 - z_2)^2}\mathcal{E}(s)|_{s=-\mu-H},$$

where $\mathcal{E}(s)$ (2.16) is the generating functions of the s-variables. Combining this fact with proposition 3.2 we have

$$u_k = -s_k - H, \qquad \dot{u}_k = -\dot{s}_k. \tag{3.21}$$

The inverse map reads as

$$s_k = -u_k + 2\mathcal{H} - \lambda^2, \qquad \dot{s}_k = -\dot{u}_k \tag{3.22}$$

Here $\dot{s}_k = \{H, s_k\}_1$ and $\dot{u}_k = \{\mathcal{H}, u_k\}_2$ and $\{\}_{1,2}$ mean the Poisson brackets (1.1) on M and the Poisson brackets (3.1) on $\mathcal{M}$, respectively.

Lemma 5. *Mappings (3.21) and (3.22) define one-to-one correspondence between solutions of equations (3.9) for the Clebsch system and solutions of equations (2.18) for the Kowalevski gyrostat on the corresponding to (3.19), (3.20) and (3.18) level surfaces of integrals.*

Proof. It is easy to see that substituting (3.18),(3.19) and (3.21) into equation (3.9) one gets equation (2.18). □

Taking into account lemmas 3, 4 and 5 we arrive at one of the main results of the paper:

Theorem 4. *Solutions of the Clebsch problem give rise to solutions of the Kowalevski gyrostat problem and vice versa.*

Proof. The map $\{\boldsymbol{x}, \boldsymbol{J}\} \to \{\boldsymbol{p}, \boldsymbol{l}\}$ (3.13)–(3.16) allows us to construct solutions of the Clebsch problem starting with the solutions of the Kowalevski gyrostat problem. □

Moreover, on the level surfaces of integrals of motion we can decompose this map $\{\boldsymbol{x}, \boldsymbol{J}\} \to \{\boldsymbol{p}, \boldsymbol{l}\}$ in the following way:

$$\{\boldsymbol{x}, \boldsymbol{J}\} \to \{s_{1,2}, \dot{s}_{1,2}\} \to \{u_{1,2}, \dot{u}_{1,2}\} \to \{\boldsymbol{p}, \boldsymbol{l}\}.$$

According to the results of lemmas 3, 4 and 5 and fixed inequalities between $u_{1,2}$ and $s_{1,2}$ (3.8), (2.14) each of the intermediate mappings is a one-to-one correspondence. So, there is the inverse map $\{\boldsymbol{p}, \boldsymbol{l}\} \to \{\boldsymbol{x}, \boldsymbol{J}\}$. We cannot rewrite this transformation in a compact form because the map $\{s_{1,2}, \dot{s}_{1,2}\} \to \{\boldsymbol{x}, \boldsymbol{J}\}$ is much more complicated than the map $\{u_{1,2}, \dot{u}_{1,2}\} \to \{\boldsymbol{p}, \boldsymbol{l}\}$ (3.12) (see [13]).

Thus we establish isomorphism of the spaces of the solutions for the Clebsch problem and the Kowalevski gyrostat problem. It means that we can construct solutions of the Kowalevski gyrostat problem using a known solution of the Clebsch system obtained either by Kobb–Kharlamova [7, 6] or by Kötter [12].

We have to underline that mappings (3.21), (3.22) and hence (3.13), (3.16) define trajectory equivalence of the Clebsch system and of the Kowalevski gyrostat on the level surfaces of integrals of motion (3.19), (3.20) and (3.18) . Of course, these mappings do not preserve the Poisson structure of the corresponding phase spaces.

4. Conclusion

The established trajectory isomorphism between proper parametrized solutions of the Kowalevski gyrostat and the Clebsch system allowed us to interpret Kowalevski variables as elliptic coordinates on the 2-sphere for the Clebsch system. We hope that the traced correspondence between two systems will help us to construct solution of the gyrostat problem using various known solutions of the Clebsch model.

A similar correspondence may be obtained for the Kowalevski gyrostat on $so(4)$ algebra, which is equivalent to the generalized Kowalevski gyrostat on $e(3)$ [11]. For $so(4)$ top initial vector $\boldsymbol{l}_{so(4)}$ was obtained in [10]. In the gyrostat case we have to substitute it by rule (3.16).

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